

## GENERAL PROBLEMS OF METROLOGY AND MEASUREMENT TECHNIQUE

### THE STRIP METHOD OF TRANSFORMING SIGNALS CONTAINING REDUNDANCY

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*The strip method of transforming signals is considered, which involves introducing redundancy with rectangular matrices in forward transformation and pseudoinverse ones in restoring the signals at the receiving end. This redundancy is used to reduce the interference power and to observe, localize, and correct noise.*

**Key words:** *strip signal transformation method, information redundancy, corruption power, pulse interference observation, localization, and correction.*

A classical task in communication theory is dealing with various forms of interference in the transmission channel that distort the signals. The methods are substantially dependent on the form of the interference. The usual principle for raising the noise immunity in the data transmission system is the introduction of distortions into the transmitted signal and reverse transformation on reception. Linear prediction methods are very numerous, and the amplitude-frequency one is the most simple. At the transmitting point, the transmitting system becomes a predicting four-terminal network in the form of a filter whose characteristic is such that at the receiving point in the channel having noise interference one can improve the signal-noise ratio by selecting the transfer coefficient of a correcting four-terminal network. Similar considerations apply in phase-frequency and amplitude-phase methods. In the case of digital signals, wide use is made of linear codes in which the encoding and decoding are based on linear matrix transformation.

In [1], a method is described for dealing with pulse noise that allows one to minimize the Chebyshev norm for the noise in the received signal. It is based on preliminary linear transformation (encoding) of the analog signal  $x(t)$  before transmission over the communication channel by dissecting the signal into parts equal in length and generating linear combinations of these parts with chosen weighting coefficients and subsequently joining together the transformed parts into one signal  $y(t)$  of the initial length. The total length of the signal is not altered, but each of the parts of the transformed signal  $y(t)$  bears information about the entire initial signal  $x(t)$ .

That transformation is based on a strip operator, and the corresponding method for transforming and restoring the signals has been called the strip method.

At the receiving end, the signal is subject to inverse strip transformation (decoding), which restores the initial signal form. From the mathematical viewpoint, the operator  $\Phi$  for transforming the signal at the transmitting end and the inverse operator  $\Phi^{-1}$  for restoring the signal at the receiving end are described by

$$\Phi = S^{-1}AS, \quad \Phi^{-1} = S^{-1}A^{-1}S, \quad (1)$$

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where  $S$  is the strip operator that transforms the initial signal of length  $T$  into an  $n$ -dimensional vector function of length  $T/n$ ;  $S^{-1}$  is the inverse operator;  $A$  is a constant nondegenerate  $n \times n$  matrix, whose elements are the coefficients for the linear combinations of the parts of the transformed signal; and  $A^{-1}$  is the inverse transformation matrix.

The following chain of equations gives a more detailed description of the procedure for encoding, transmitting, and restoring the signal:

$$\begin{aligned} X &= Sx, & Y &= AX, & y &= S^{-1}Y, & y' &= y + n, \\ Y' &= Sy', & X' &= A^{-1}Y', & x' &= S^{-1}X'. \end{aligned} \quad (2)$$

Here  $x(t)$  denotes the initial signal of duration  $T$ ;  $y(t)$  is the signal of duration  $T$  transmitted over the channel, while  $y'(t)$  is the sum of the signal  $y(t)$  and the noise  $n(t)$  at the output of the communication channel; and  $x'(t)$  is the restored signal of duration  $T$ .

The strip operator  $S$  is equivalent to splitting up a long initial signal  $x(t)$ ,  $0 \leq t \leq T$  into  $n$  parts of equal duration  $h = T/n$  and getting  $n$  short signals of the form

$$x_1(t) = x(t), x_2(t) = x(t + h), \dots, x_n(t) = x(t + (n - 1)h), \quad 0 \leq t \leq h.$$

From these we generate the  $n$ -dimensional vector function

$$X(t) = \begin{bmatrix} x_1(t) \\ \dots \\ x_n(t) \end{bmatrix}, \quad 0 \leq t \leq h.$$

We use a nonsingular square matrix  $A = [a_{ij}]_{1,n}$  whose elements are real numbers to transform the vector  $X(t)$  to the vector

$$Y(t) = AX(t) = \begin{bmatrix} y_1(t) \\ \dots \\ y_n(t) \end{bmatrix}, \quad 0 \leq t \leq h.$$

The components of  $Y(t)$  are defined by  $y_i(t) = A_i X(t)$ ,  $i = 1, 2, \dots, n$ , in which  $A_i$  is row  $i$  of matrix  $A$ .

The operator  $S^{-1}$  is inverse to the operator  $S$  and performs the operator of joining together the signals  $y_i(t)$ ,  $0 \leq t \leq h$ ,  $i = 1, 2, \dots, n$ , into a single signal  $y(t)$  of duration  $T$ .

This completes the encoding procedure for the initial signal; then  $y(t)$  is transmitted over the communication channel containing noise and at the receiving end is subject to a decoding procedure, which uses the matrix  $A^{-1}$ .

It has been shown [1, 2] that the strip method raises the noise immunity in signal transmission in relation to pulse noise. This occurs because a uniform distribution is obtained for the pulse noise over the duration of the output signal  $x(t)$  (noise stretching). The greatest effect is attained if as matrix  $A$  we used normalized Hadamard matrices; then if the power in the pulse noise is maintained, the amplitude can be reduced by a factor  $\sqrt{n}$ .

These results apply to square matrices  $A$ . Here the strip method is used for rectangular matrices  $A$  in which there are more rows than columns. Then the duration of the transformed signal will be greater than that of the initial signal:  $T' > T$ , which means that information redundancy is introduced. The redundancy can be used firstly to reduce the mean interference power in restored signal  $x'(t)$  and secondly to observe, localize, identify, and correct the noise. We now examine both of these tasks.

**Reducing Noise Power in the Recovered Signal.** Information redundancy is widely used in communication theory for dealing with noise. The theory of the least-squares method shows that averaging  $n$  measurements reduces the error by a factor  $\sqrt{n}$ . In the strip method, with redundancy in matrix  $A$  in transformations (1) and (2), that matrix has dimensions  $m \times n$ , where  $m > n$ . One-dimensional signal transformation gives

$$y(t) = y_1(t) + y_2(t - h) + \dots + y_m(t - (m - 1)h), \quad 0 \leq t \leq T' \quad (3)$$

with duration  $T' = mn > T$  and contains  $k = m - n$  redundant parts.

Now not all the rows of matrix  $A$  will be linearly independent;  $k$  rows can always be represented as a linear combination of the others. The inverse linear transformation operator in that case describes the pseudoinverse matrix  $A^+$  of dimensions  $n \times m$  [3]:

$$A^+ = (A^T A)^{-1} A^T.$$

Using  $A^+$  is equivalent to using least-squares fitting and leads to the maximum reduction for a normal distribution of noise at a given matrix  $A$  in the standard deviation of the restored signal  $x'(t)$ .

The total noise level is reduced by a factor  $[(n+k)/n]^{1/2} = (m/n)^{1/2}$ ; the operator described by matrix  $A^+$  enables one to reduce the power of any noise, including noise other than pulsed, provided it is centered.

As an example, we consider a rectangular matrix  $A$  of dimensions  $12 \times 4$ :

$$A = \frac{1}{\sqrt{12}} \begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \end{bmatrix}^T.$$

It is derived from a Hadamard matrix of order  $n = 12$  written in normal form by isolating the first four columns. Then the number of redundant parts  $k = m - n = 8$ , and the transformed signal  $y(t)$  has a total duration  $T' = 3T$  that is three times larger than that of the initial signal  $x(t)$ . The recovered signal  $x'(t)$  derived by means of pseudoinverse matrix  $A^+$  (in the present case  $A^+ = A^T$ ) has duration  $T$ . The improvement in noise immunity by comparison with nonredundant transformation is by a factor  $[(n+k)/n]^{1/2} \approx 1.7$ .

**Estimating Redundancy for Correcting Pulse Noise.** I consider using redundancy for the case of pulse noise on video signal transmission. The reasons for the occurrence of such noise are often communication interruptions, fading in the channel, and so on, which lead to brief dropout in the signal. Pulse noise is encountered for example in recording a TV image signal on a video tape recorder. Here the dropout is due to deviation from contact between the magnetic head and the tape on account of defectiveness, cemented patches, and the dynamics of the vacuum contact [4].

Interference of signal dropout type in essence alters the signal amplitude down to zero. In dealing with this, it is best to use codes providing error detection and correction, only not for digital signals but instead for analog ones. For the transmission of a signal with information redundancy, this means determining the corrupted part of the signal and restoring it on the basis of the redundant information.

Redundancy in accordance with (3) enables one to detect the fact of pulse noise (observation task), determine the time position, i.e., the number of the corrupted signal part (diagnosis task), and also the amplitude and waveform of the interference (identification task). This allows one to handle the correction task (eliminating, correcting, or excluding the noise).

Consider the lower bound to the redundancy necessary for observing and correcting pulse noise. The basis is provided not by a particular correction algorithm but instead by general considerations from information theory. We consider signals of duration  $T$  conditionally divided into  $n$  parts of length  $h$ . Introducing redundancy causes an increase in the signal duration by  $\Theta$ , so the length of the transformed signal is  $T' = T + \Theta$ .

We assume that signal dropout occurs fairly rarely, not more than  $r$  times in the interval  $T'$  ( $r$ -fold noise), while the duration of each dropout does not exceed  $h$ . The volume of added redundancy is determined as the relative increase in signal duration  $R = \Theta/T$  on encoding (forward strip transformation). We denote by  $R_{\text{ob}}$ ,  $R_{\text{loc}}$ , and  $R_{\text{cor}}$  the minimal volumes of redundancy required for observing, localizing, and correcting pulse noise.

*Theorem.* The minimum redundancy volume needed for localizing and correcting  $r$ -fold pulse noise when each instance lasts not more than  $h = T/n$  constitutes  $R = (r+1)/n$ .

To prove this, we start from the amount of information on the noise it is desirable to obtain. To communicate on the presence of the noise requires minimal information of yes-no type at the limit equal to one bit.

When one determines the number of the corrupted part, if the noise is signal, the required amount of information is  $\log_2 n$ , where  $n$  is the number of parts. For  $r$ -fold noise, this increases to  $\log_2 \sum_{i=1}^r C_n^i$ , where  $C_n^i$  is the number of combina-

tions of  $n$  items taken  $i$  at a time. In both cases, the volume of information borne by one redundant part of the analog signal is certainly larger than this, so the minimal redundancy volume constitutes  $R_{\text{ob}} = R_{\text{loc}} = n^{-1}$ .

To correct  $r$ -fold noise, it is necessary to know not only the numbers of the corrupted parts (this requires increasing the signal duration by  $h$ ) but also the correct values of the signal in the  $r$  corrupted parts (this requires additional increase in the signal duration by  $rh$ ). Then

$$R_{\text{cor}} = (r + 1)/n. \quad (4)$$

It follows from (4) that  $r$ -fold noise correction requires the initial signal containing  $n$  parts to be transformed to a signal containing  $n + r + 1$  parts. This corresponds to increasing the length of the signal by  $\Theta = (r + 1)h$ , i.e., introducing redundancy for volume  $(r + 1)/n$ .

For single noise, (4) becomes  $R_{\text{cor}} = 2/n$ , i.e., to observe and correct single noise alteration  $\tau \leq h$  requires the addition of two parts of length  $h$  to the initial signal.

The general aspects of encoding theory [5] imply that to correct errors of order  $r$  requires not less than  $2r$  redundant components, i.e.,  $k \geq 2r$ . The present estimate is much more economical:  $k \geq r + 1$ , and it coincides with the estimate in the redundant-variable method [6], which implies that on processing analog information it is sufficient to have  $k = r + 1$  redundant components in order to determine the place and amplitude of practically all  $r$ -fold occurrences of noise.

**Noise Localization and Correction.** The (4) estimate indicates the minimum redundancy necessary to correct  $r$ -fold noise. To show that this value is sufficient, one needs to propose detailed correction algorithms. One of these is described below.

Let the rectangular matrix  $A$  in (1) and (2) have dimensions  $m \times n$ , where  $m > n$ ; the number  $k = m - n$  characterizes the number of additional parts of length  $h$  in the signal  $y(t)$  by comparison with  $x(t)$ . As any  $k$  parts of  $y(t)$  are linear combinations of the others, we have

$$\Delta_i = \sum_{j=1}^{n+k} b_{ij} y_j = 0, \quad i = 1, 2, \dots, k, \quad (5)$$

in which  $b_{ij}$  are certain numerical coefficients.

When noise occurs, these error signals  $\Delta_i$  become different from zero and contain certain information about the place and amplitude of the noise.

We consider the case of single noise  $r = 1$  (one part of the signal corrupted), and then  $k = r + 1 = 2$ . We generate the two error signals  $\Delta_1$  and  $\Delta_2$  of (5) and form  $n + 2$  linear combinations of them:

$$\Delta'_i = b_{2i} \Delta_1 - b_{1i} \Delta_2 = \begin{vmatrix} \Delta_1 & \Delta_2 \\ b_{1i} & b_{2i} \end{vmatrix}, \quad i = 1, 2, \dots, n + 2. \quad (6)$$

It is readily shown that each of the errors  $\Delta'_i$  will be invariant to distortion in one of the parts of  $y(t)$ . In fact, if the result of noise corruption is that the signal  $y_j$  becomes  $y_j + \delta y_j$ , then we have

$$\Delta_1 = -b_{1j} \delta y_j; \quad \Delta_2 = -b_{2j} \delta y_j. \quad (7)$$

We substitute these into (6) to get

$$\Delta'_i = b y_j \begin{vmatrix} b_{1j} & b_{2j} \\ b_{1i} & b_{2i} \end{vmatrix}; \quad i = 1, 2, \dots, n + 2.$$

This implies that if none of the second-order determinants compiled from the  $b_{ij}$  is zero,

$$\begin{vmatrix} b_{1j} & b_{2j} \\ b_{1i} & b_{2i} \end{vmatrix} \neq 0; \quad i \neq j,$$

then all the error signals apart from one  $\Delta'_i$  will be different from zero. This serves to define the number of the corrupted part. The noise amplitude is given by (7).

I now consider the particular case where the transformed signal is obtained by adding two redundant parts to the initial signal. Matrix  $A$  is then obtained by adding two rows to unit matrix:

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{bmatrix}. \quad (8)$$

Here the initial signal remains unaltered, so the forward and reverse transformations are considerably simplified. The additional parts are generated in accordance with

$$y_{n+1}(t) = \sum_{j=1}^n a_j x_j(t), \quad y_{n+2}(t) = \sum_{j=1}^n b_j x_j(t).$$

As in the present case  $x_j(t) = y_j(t)$ ;  $j = 1, 2, \dots, n$ , (5) becomes

$$\Delta_1 = a_1 y_1 + a_2 y_2 + \dots + a_n y_n - y_{n+1} = 0;$$

$$\Delta_2 = b_1 y_1 + b_2 y_2 + \dots + b_n y_n - y_{n+2} = 0.$$

The coefficients  $a_j$  and  $b_j$  may be chosen for example as follows:

$$a_j = -1; \quad b_j = -j; \quad j = 1, 2, \dots, n. \quad (9)$$

Then we get for the error signals that

$$-\Delta_1 = y_1 + y_2 + \dots + y_n + y_{n+1} = 0;$$

$$-\Delta_2 = y_1 + 2y_2 + \dots + ny_n + y_{n+2} = 0.$$

The number of the part corrupted by noise is defined by the ratio of the signals  $N = \Delta_2/\Delta_1$ ; to avoid possible division by zero, it is better to write this in the form

$$\Delta_2 = N\Delta_1. \quad (10)$$

In the  $(\Delta_1, \Delta_2)$  plane, this defines  $n$  straight lines passing through the origin, where  $N$  is the slope. For example, if the first part is corrupted we get  $\Delta_1 = \Delta_2 = \delta y_1$ ,  $N = 1$ ; if the second part is corrupted,  $\Delta_1 = \delta y_2$ ,  $\Delta_2 = 2\delta y_2$ ,  $N = 2$ , and so on up to part  $n$ . To correct the noise, it is sufficient to subtract from the corrupted section, whose number has already been determined, the error signal  $\Delta_1$ , since in the case of a single error it coincides with the noise:  $\Delta_1 = \delta y_N$ .

**Computer Simulation Results.** To test the strip method and model the algorithm, I used the MATLAB package [7], in which in particular there is a command *strips*, whose action is analogous to that of the strip operator. It outputs to the screen a long graph of the function divided into parts. Unfortunately, the result from this command is inaccessible in the subsequent processing, so to realize the strip operator it is necessary to write a user  $m$  function.

I now consider an example of generating a redundant signal by means of the (8) matrix and its subsequent processing with the normalized coefficients of (9). The calculations are performed in the MATLAB package by means of the *redund* program.

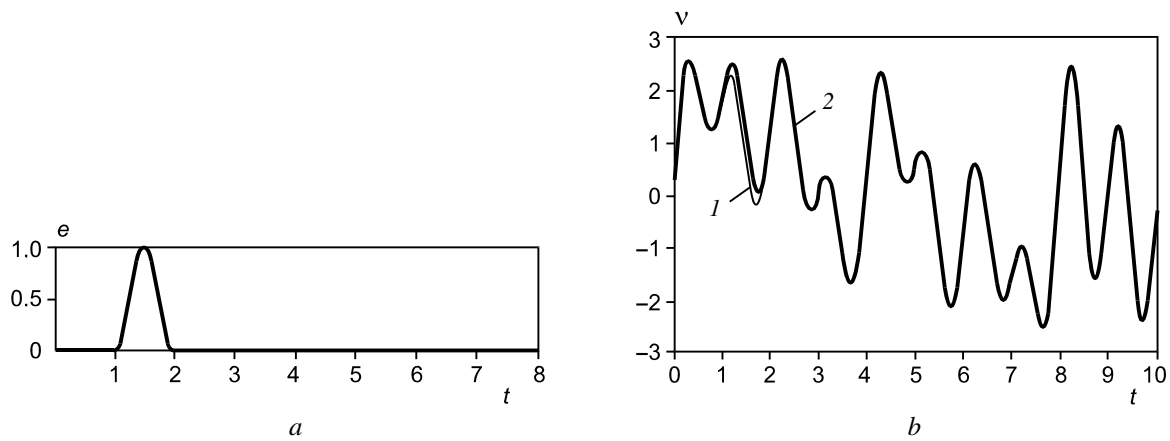


Fig. 1. Pulse noise (a) and redundant signal (b): 1) initial signal; 2) signal with corruption in the second part.

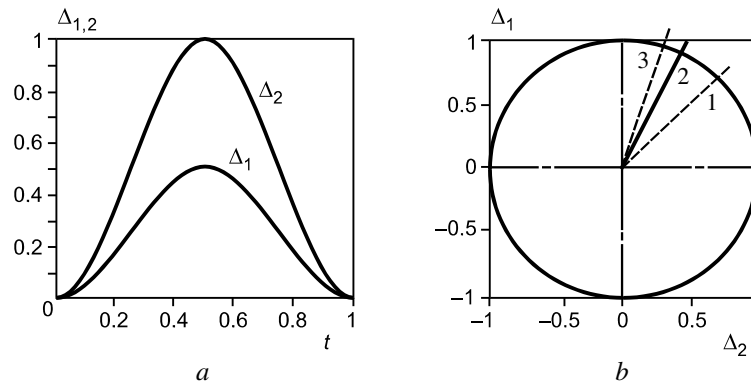


Fig. 2. Control signals (a) and error plane (b).

**Example 1.** The initial signal  $x(t)$  is obtained by summing five sinusoidal components:

$$x(t) = \sin 2\pi t + \sin \pi t + \sin \frac{\pi}{2}t + \sin \frac{\pi}{4}t + 0.3 \sin \frac{\pi}{8}t, \quad 0 \leq t \leq 8.$$

The strip transformation of this is obtained by dissecting it into eight equal parts  $y_1, \dots, y_8$  of length 1 sec each. The redundant parts are obtained from

$$y_9 = \frac{1}{4} \sum_{i=1}^8 y_i,$$

$$y_{10} = (y_1 + 2y_2 + 3y_3 + 4y_4 + 5y_5 + 6y_6 + 7y_7 + 8y_8)/20.$$

Then from the short signals  $y_1, \dots, y_{10}$  we generate a redundant scalar signal  $y(t)$  of length 10 sec and to it add noise that corrupts the second part. The waveform of this noise is shown in Fig. 1a; the redundant signals with the noise 2 and without it 1 are shown in Fig. 1b (noise added with factor 0.5). Figure 1b shows that the noise slightly corrupts the transmitted signal and to observe the effect by eye is extremely difficult.

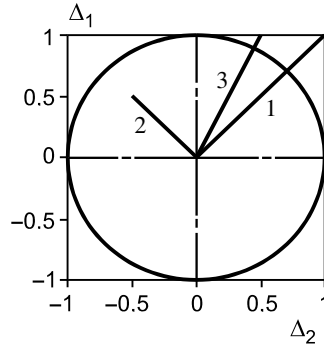


Fig. 3. Straight lines in the error plane.

To determine the number of the corrupted part and the corruption there, we derive the control signals (error signals)

$$\Delta_1 = \sum_{i=1}^8 y_i - 4y_9; \quad \Delta_2 = \sum_{i=1}^8 iy_i - 20y_{10}. \quad (11)$$

Figure 2a shows these control signals as time functions; their shape repeats that of the noise (Fig. 1a), while the amplitude ratio is 2. This means that the number of the error part is  $N = 2$ , which is correct.

The same conclusion can be drawn from Fig. 2b, which shows the  $(\Delta_1, \Delta_2)$  plane; slope of a straight line is 2, which again confirms that  $N = 2$ . The dashed line in this in each case shows the direction of the straight line with corruption in the first or third parts, i.e., for noise with  $N = 1$  or  $N = 3$ . In view of the small angle between the lines 2 and 3, the exact noise diagnosis may be difficult. It is even more difficult to distinguish noise with large numbers, e.g.,  $N = 7$  and  $N = 8$  because the choice of  $a_i$  and  $b_i$  in (9) is not the best, as it does not incorporate the following factors. Firstly, to obtain power identical on average in each part it is necessary that the norms of all the rows in matrix  $A$  should be one. Secondly, one should provide the maximum angular distance between the straight lines in (10). To incorporate the first factor, the normalizing factors of  $1/4$  and  $1/20$  have been introduced in generating parts  $y_9$  and  $y_{10}$ . As for the second, one must change  $a_i$  and  $b_i$  by comparison with (9). One of the cases of such change is given below.

**Example 2.** We consider the same initial signal as in the previous example, but with the control signals now obtained from

$$\Delta_1 = y_1 - y_2 + y_3 - y_4 + y_5 - y_6 + y_7 - y_8 - 4y_9;$$

$$\Delta_2 = (y_1 + y_2) + 2(y_3 + y_4) + 3(y_5 + y_6) + 4(y_7 + y_8) + 10y_{10}.$$

By comparison with (11), the coefficients in the first control condition are taken with different signs, while the variables in the second control condition are grouped in pairs. The graphs for these control signals with the noise shown in Fig. 1a differ only in the signs of  $\Delta_1(t) = -\Delta_2(t)$ , which implies that the noise has corrupted the second part.

In the case of corruption in the third part, the control signals satisfy  $\Delta_2 = 2\Delta_1$ ; Fig. 3 implies that the angular distances between the straight lines in the error plane are increased by comparison with Fig. 2b from the previous example, which improves the distinguishability of single noise, i.e., improve diagnosis.

The best from the viewpoint of distinguishing single noise is a uniform distribution of the straight lines in the quadrants in the  $(\Delta_1, \Delta_2)$  plane; this is provided with the following values for the coefficients:

$$b_{1j} = \cos \frac{\pi j}{n+2}, \quad b_{2j} = \sin \frac{\pi j}{n+2}.$$

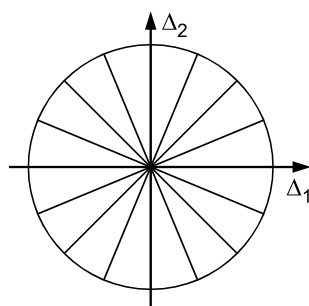


Fig. 4. Uniform disposition of straight lines for  $n = 6$ .

This provides equal angles between adjacent straight lines. Figure 4 shows an example of this for  $n = 6$ ; the directions of the  $n + 2$  straight lines are specified by the diagonals of a regular octagon. A feature of one of the eight possible single-noise processes will be the location of the  $(\Delta_1, \Delta_2)$  image point on the corresponding straight line.

**Conclusions.** The strip method is an effective means of dealing with pulse noise; it introduces redundances into the transmitted message, which can be used not only to reduce the noise power in the received signal but also to observe, localize, and correct it. This leads to a substantial reduction in the error in transmitting the signals over transmission channels. An analogous procedure can be used for the noise-immune transmission of two-dimensional signals (images).

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